Figuring the Acceleration of the Simple Pendulum

Martin Lieberherr, Mathematisch Naturwissenschaftliches Gymnasium Rämibühl, Physikalisches Institut, Zürich, Switzerland

The centripetal acceleration has been known since Huygens' (1659) and Newton's (1684) time.[1,2] The physics to calculate the acceleration of a simple pendulum has been around for more than 300 years, and a fairly complete treatise has been given by C. Schwarz in this journal.[3] But sentences like "the acceleration is always directed towards the equilibrium position" beside the picture of a swing on a circular arc can still be found in textbooks, as e.g. in Ref. 4. Vectors have been invented by Grassmann (1844)[5] and are conveniently used to describe the acceleration in curved orbits, but acceleration is more often treated as a scalar with or without sign, as the words *acceleration/deceleration* suggest. The component tangential to the orbit is enough to deduce the period of the simple pendulum, but it is not enough to discuss the forces on the pendulum, as has been pointed out by Santos-Benito and A. Gras-Martí.[6] A suitable way to address this problem is a nice figure with a catch for classroom discussions or homework. When I plotted the acceleration vectors of the simple pendulum in their proper positions, pictures as in Fig. 1 appeared on the screen. The endpoints of the acceleration vectors, if properly scaled, seemed to lie on a curve with a familiar shape: a cardioid. Is this true or just an illusion?

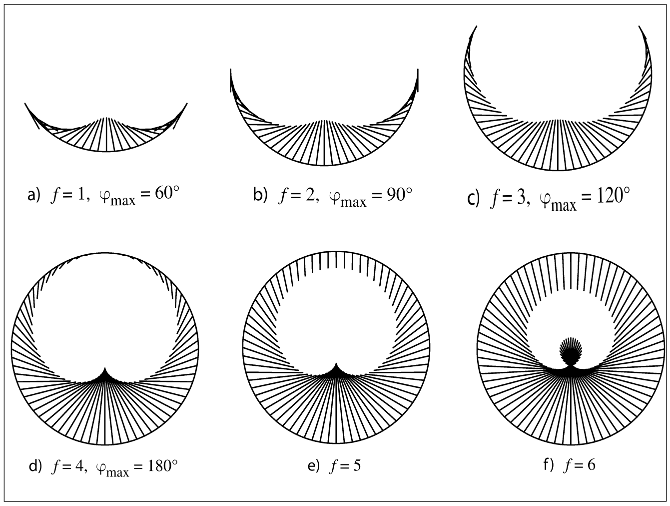

Fig. 1. Acceleration vectors (inward pointing lines) of the simple pendulum drawn at regular intervals. The speed at the lowest point is $v_0 = \sqrt{fgr}$.

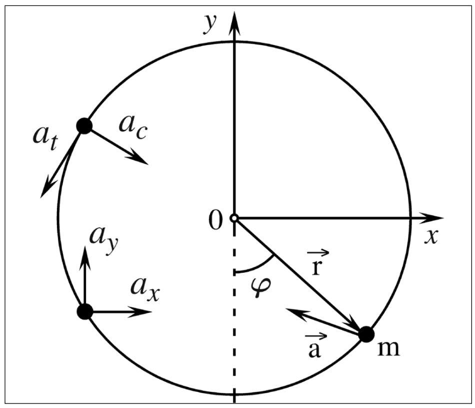

Fig. 2. Simple pendulum of length *r* with components of acceleration (not to scale).

Let v be the speed of the bob of a simple pendulum of fixed length r, as seen in Fig. 2. The acceleration has centripetal and tangential components:

$$a_c = v^2/r \tag{1}$$

$$a_t = g\sin\varphi. \tag{2}$$

The speed at angle φ can be calculated from the speed v_0 at the lowest point:

$$v^2 = v_0^2 - 2gr(1-\cos\varphi). \tag{3}$$

The bob cannot reach the top if $v_0^2 < 4gr$ and a string pendulum will not describe a full circle unless $v_0^2 \geq 5gr$. A dimensionless factor f is defined as:

$$f = \frac{v_0^2}{gr}. \tag{4}$$

With this convention, the centripetal acceleration is

$$a_c = (f-2)g + 2g\cos\varphi \tag{5}$$

and the amplitude of the oscillation is

$$\varphi_{max} = \arccos(1 - f/2) \quad (\text{if } f \leq 4). \tag{6}$$

To find the curve on which the endpoints of the acceleration vectors lie, it is convenient to express the acceleration in x- and y-components, as seen in Fig. 2:

$$\begin{aligned} a_x &= -a_c\sin\varphi - a_t\cos\varphi \\ &= -[(f-2)g + 2g\cos\varphi]\sin\varphi - g\sin\varphi\cos\varphi \end{aligned} \tag{7a}$$

$$\begin{aligned} a_y &= a_c\cos\varphi - a_t\sin\varphi \\ &= (f-2)g\cos\varphi + 3g\cos^2\varphi - g. \end{aligned} \tag{7b}$$

The center of the circular orbit of the pendulum bob is the origin 0 of the coordinate system. The vector sum $\mathbf{r} + \mathbf{a}\cdot s/g$ with parameter s represents a straight line through the bob in

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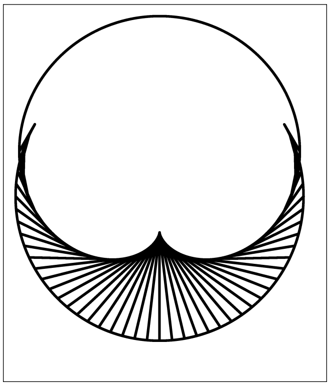

Fig. 3. ***f*** **= 3,** $\varphi_{\max}$ **= 120°, with cardioid. The scaling constant in Eq. (8) is** ***s*** **=** ***r*****/4.**

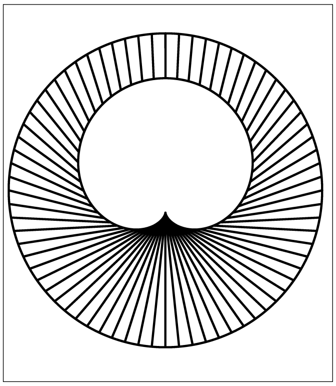

Fig. 4. ***f*** **= 6, with cardioid. The scaling constant in Eq. (8) is** ***s*** **=** ***r*****/7.**

the direction of the acceleration (see Fig. 2). The components *x* and *y* of this sum are:

$$\begin{aligned} x &= r\sin\varphi + \frac{a_x}{g}s \\ &= (r - fs + 2s)\sin\varphi - 3s\sin\varphi\cos\varphi \end{aligned} \tag{8a}$$

$$\begin{aligned} y &= -r\cos\varphi + \frac{a_y}{g}s \\ &= -(r - fs + 2s)\cos\varphi + 3s\cos^2\varphi - s. \end{aligned} \tag{8b}$$

With the abbreviations

$$p = r - fs + 2s \tag{9a}$$

$$q = 3s, \tag{9b}$$

we get

$$x = p\sin\varphi - q\sin\varphi\cos\varphi \tag{10a}$$

$$y = -p\cos\varphi + q\cos^2\varphi - s. \tag{10b}$$

Eq. (10) is the parametric representation of a curve called Pascal's limaçon.[7] For $p = q$ it is a cardioid (a one-cusped epicycloid).[8] The endpoints of the acceleration vectors will trace out a cardioid, if we choose the scaling factor s to have the value

$$s = \frac{r}{f+1}. \tag{11}$$

In Figs. 3-4, the vectors are plotted in this way, together with the cardioid.

The cardioid has a cusp. This cusp implies that the acceleration is a local or global extremum at that point. You may want to check this in Fig. 5, which summarizes the details of

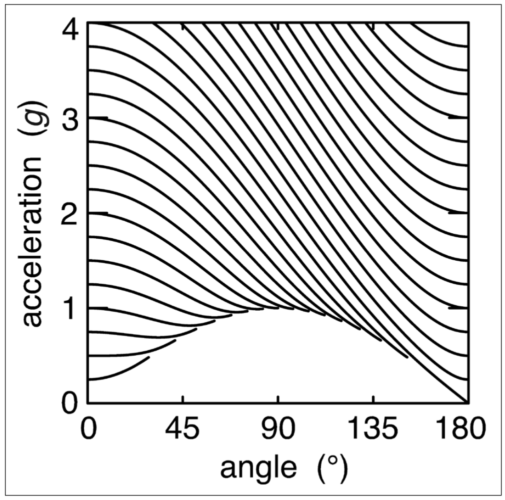

Fig. 5. Magnitude of the acceleration of a simple pendulum as a function of the angle, plotted for different initial speeds $v_0 = \sqrt{fgr}$ **at the lowest point (angle 0°). The acceleration at the lowest point is** ***fg*****.**

Ref. 3. Focusing on the curve whose intercept with the left axis is 2, we see that the acceleration at the lowest point of that orbit is 2*g* and is a maximum, and we also see that the acceleration decreases to the value *g* at its turning points of ±90°. This curve corresponds to the case plotted in Fig. 1(b).

Acknowledgments

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Mathematisch Naturwissenschaftliches Gymnasium Rämibühl, Physikalisches Institut, Zürich, Switzerland; lieberhm@mng.ch