The rocker – Experiments with an anharmonic oscillator

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1 Introduction

Every instructor should know some easy examples of anharmonic oscillations. The rocking of an empty wine bottle or a slender beer glass is one of those: The angle $\varphi(t)$ is not a sinusoidal function of time and the period is not independent of the amplitude, not even for small amplitudes. Equivalent oscillations of LEGO-rockers are experimentally investigated with simple equipment.

This kind of oscillation has been studied by G. W. Housner[1] (response of slender structures to an earthquake) and by T. McGeer[2] (as a part of bipedal robotic motion). The equation of motion has been solved by G. W. Housner.[1] T. McGeer and L. H. Palmer[2] showed that the oscillation comes to rest after a finite time. This article summarizes the theory, extends it and presents measurements of $\varphi(t)$ with consumer electronics. Rocking is one of the few anharmonic oscillations that are accessible to undergraduate students.

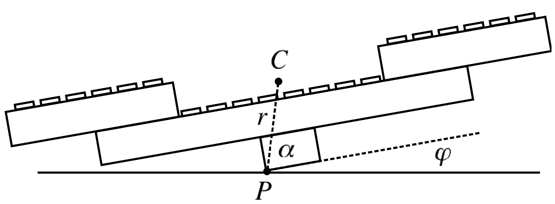

Figure 1: A rocker is built with long and short LEGO™ bricks. Additional bricks on the arms increase the period. The rocker rotates around the pivot point P. The center of mass C is at a distance r from P. The line CP includes an angle α with the bottom of the rocker and the bottom includes an angle φ with the horizontal table.

2 Theory

A rocker is built with LEGO™ bricks, see Fig. 1. It rotates around the pivot point P under the influence of gravity. The axis of rotation through P is assumed to be fixed with respect to the rocker and to the supporting table, i.e. we assume a no-slip condition and planar motion. The rocker has mass m and moment of inertia J with respect to P. The center of mass C is at a distance r from P. The rocker is left-right symmetrical and so is the movement around the left and right pivot point.[3] The line CP includes an angle α with the bottom of the rocker. The bottom includes the time dependent angle φ with the horizontal table. The equation of motion is

$$J\ddot{\varphi} + mgr\cos(\alpha + \varphi) = 0\,. \tag{1}$$

With the substitution $\beta = \alpha + \varphi - \pi/2$ we get $J\ddot{\beta} + mgr\sin\beta = 0$, the equation of motion for the physical pendulum in the usual notation. The exact solution is an elliptic integral.[4] The elliptic integral can be approximated by a sine if β is small. β is not small in our case, because α can be varied almost at will by adding bricks to the rocker. For our purpose it is easier to expand $\cos(\alpha + \varphi)$ in a Taylor series around $\varphi = 0$, because φ is kept small in our experiment.[5] Up to first order Eq. (1) becomes

$$\ddot{\varphi} + \frac{mgr}{J}\left(\cos(\alpha) - \sin(\alpha)\cdot\varphi\right) = 0\,. \tag{2}$$

The solution of Eq. (2) is a hyperbolic cosine:

$$\varphi(t) = c - a\cosh\left(\frac{t - t_0}{\tau}\right) \tag{3}$$

$$c = \frac{1}{\tan\alpha} \tag{4}$$

$$\tau = \sqrt{\frac{J}{mgr\sin\alpha}}\,. \tag{5}$$

The parameters c and τ are device specific. The free parameters a and t_0 are determined by the starting conditions, but the roots or zeros t_1 and t_2 of Eq. (3) are better suited for data analysis:

$$t_{1,2} = t_0 \pm \tau\cdot\mathrm{arcosh}(c/a) \tag{6}$$

$$t_0 = \frac{t_1 + t_2}{2} \tag{7}$$

$$\Delta t = t_2 - t_1 \tag{8}$$

$$a = \frac{c}{\cosh\left(\frac{\Delta t}{2\tau}\right)} \tag{9}$$

$$\varphi(t, t_1, t_2) = c\left(1 - \frac{\cosh\left((t - t_0)/\tau\right)}{\cosh\left(\Delta t/(2\tau)\right)}\right)\,. \tag{10}$$

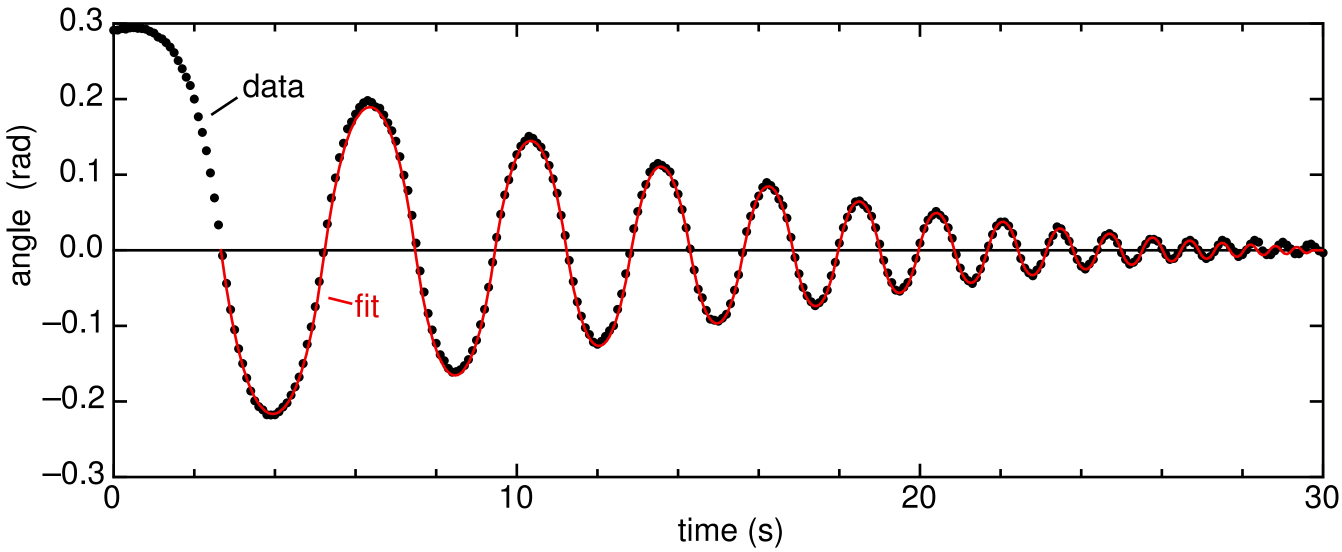

Figure 2: A LEGO-rocker oscillates for several seconds. The fit consists of 30 joined hyperbolic cosines, see Eq. (10) and (13), with a total of 5 adjustable parameters. Their values are c = 0.3395 rad, τ = 0.7691 s, t_1 = 2.663 s, t_2 = 5.239 s, and δ = 0.8744.

The amplitude $\widehat{\varphi}$ of the 'bumps' (half cycles) decreases due to damping and so does the half period Δt. How does Δt depend on $\widehat{\varphi}$? For $t = t_0$ we get from Eq. (10)

$$\widehat{\varphi} = c\left(1 - \frac{1}{\cosh\left(\frac{\Delta t}{2\tau}\right)}\right). \tag{11}$$

Solving Eq. (11) for Δt gives

$$\Delta t = 2\tau \cdot \operatorname{arcosh}\left(\frac{1}{1-\widehat{\varphi}/c}\right). \tag{12}$$

For very small amplitudes $\Delta t \propto \sqrt{\widehat{\varphi}}$, i.e. the rocking frequency rises approximately like the reciprocal square root of the amplitude.

3 Experiment

The bottles and glasses mentioned in the introduction are useful as quick demonstrations but difficult to experiment with. Care has to be taken that they do not slip or rotate around a vertical axis.[6] Because they usually have rounded bottom edges, the pivot point is not fixed. To avoid these difficulties, simple devices have been built with LEGO$^{\text{TM}}$ bricks, see Fig. 1. The advantage of the LEGO-rockers is that additional bricks on the arms increase the moment of inertia and subsequently the period of the oscillation. Bricks above or below the arms rise or lower the center of mass. Small or large (DUPLO) bricks have been used. Movies have been taken with a digital photographic camera at 15 frames per second (fps), with a mobile phone at 30 fps or with a high speed camera at 50 fps. The type of camera is not critical. The videos have been analyzed with Logger Pro.[7] The angles $\varphi(t)$ have been compared to theory with pro Fit.[8] The high speed camera movies have been used for the figures in this text.

4 Results

Some ten bumps are visible, see Fig. 2. If a hyperbolic cosine as in Eq. (3) or (10) is fitted to one bump, see Fig. 3, the residuals are of the same order of magnitude as the measurement accuracy, about 2 mrad, and are randomly distributed around zero. A sine or a parabola fits distinctly worse, see the residuals in Fig. 4. If the sine is offset, $\varphi(t) = asin(\omega(t - t_0)) + b$, to match the number of parameters with the hyperbolic cosine, the result is nearly identical to a parabolic fit. If the supporting table is slightly inclined, the bumps for positive and negative angles have different amplitudes. [2] This was never a problem in our experiments.

Fig. 2 displays a damped oscillation, but Fig. 3 shows symmetrical bumps. This means that energy is pre-

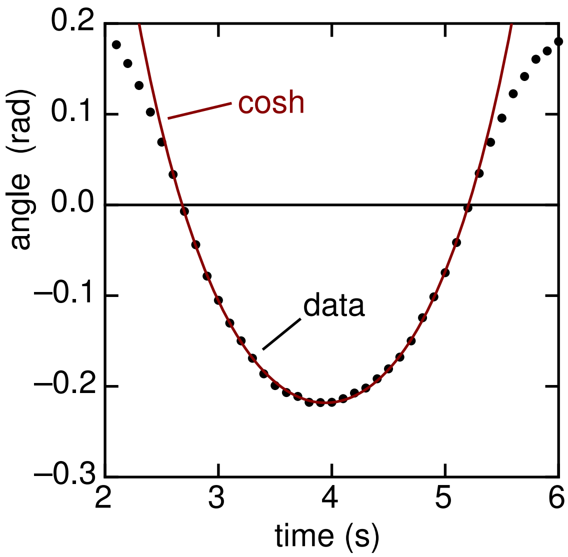

Figure 3: A hyperbolic cosine is fitted to one bump of the rocking motion (negative angles). The fitted function, see Eq. (10), has 4 adjustable parameters. The residuals are shown in Fig. 4. The symmetry of fit and data shows that friction is negligible during rotation.

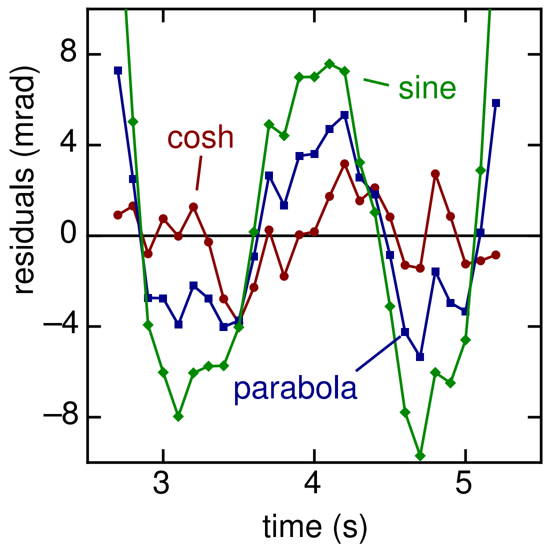

Figure 4: Residuals of the fit with a hyperbolic cosine, see Fig. 3, a parabola and a sine. The cosh-function clearly fits best. Measurement accuracy is about 2 mrad.

dominantly lost when the rocker hits the table and not during the rotation.[1, 2] By analogy with the coefficient of restitution of e.g. a bouncing ball, we would expect the amplitude $\widehat{\varphi}$ to decrease by a constant factor δ from bump to bump, i.e. $\hat{\varphi}_{i+1} = \hat{\varphi}_i \cdot \delta$. If we know this factor, the trailing bumps are determined by the leading one.

Several bumps are fitted at the same time by joining functions of the type of Eq. (10) continuously at their roots. The fitted function is

$$\begin{aligned} \varphi(t) = \quad & \varphi(t, t_1, t_2) \cdot (t_1 \leq t \wedge t < t_2) \\ + \quad & \varphi(t, t_2, t_3) \cdot (t_2 \leq t \wedge t < t_3) \\ + \quad & \ldots , \end{aligned} \tag{13}$$

where the boolean expression $(t_1 \leq t \wedge t < t_2)$ is 1 if true and 0 if false. Appropriate starting values for the zeros t_1, t_2 and the global parameters c, τ have to be supplied. $\widehat{\varphi}$ is calculated with Eq. (11) and multiplied by δ to give the amplitude of the next bump. Then Eq. (12) is used to calculate Δt of this bump and its zeroes t_2 and $t_3 = t_2 + \Delta t$, and so on. The fit in Fig. 2 was calculated in this way, i.e. the five parameters c, τ, t_1, t_2, and δ are enough to fit all 30 bumps. The fit must have one parameter more than a damped sinusoidal oscillation because the hyperbolic cosines need to be vertically shifted. The parameters in Fig. 2 have been optimized with pro Fit, but this can also be done by trial and error. A value of $c = 0.3395$ (rad) translates to $\alpha \approx 70°$, about compatible to Fig. 1.

5 Conclusion

For small amplitudes, the angular orientation $\varphi(t)$ of a rocker is a piecewise sum of hyperbolic cosines. The rocking frequency rises as the amplitude decreases. The experiment can be done with equipment that a typical student has at home: a digital camera/mobile phone and a computer.

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References

[1] G. W. Housner, "The behavior of inverted pendulum structures during earthquakes," Bull. Seismological Soc. of Am. **53**(2), 403–417 (1963).

[2] T. McGeer, and L. H. Palmer, "Wobbling, toppling, and forces of contact," Am. J. Phys. **57**(12), 1089–1098 (1989).

[3] The theory is easily extended to asymmetrical rockers.

[4] F. M. S. Lima, “Analytical study of the critical behavior of the nonlinear pendulum,” Am. J. Phys. **78**(11), 1146–1151 (2010).

[5] G. W. Housner, T. McGeer, and L. H. Palmer treat rocking of slender objects introducing approximations for $\cos(\alpha)$ and $\sin(\alpha)$. This approximation is not possible for the LEGO-rocker.

[6] To experiment with a bottle, place it on a sheet of paper (to prevent slipping) and take a movie in top view (so a slight rotation does not hinder measurement of $\varphi(t)$).

[7] available at ⟨http://www.vernier.com⟩.

[8] available at ⟨http://www.quansoft.com⟩.