

# The magnetic field lines of a helical coil are not simple loops

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It is shown that the magnetic field lines of a loosely wound helical coil of finite length do not close after one loop. Examples of simple wire shapes that display the same effect are given. © 2010 American Association of Physics Teachers.

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## I. INTRODUCTION

By serendipity I stumbled upon a common textbook error.<sup>1-5</sup> When I plotted the magnetic field lines of a helically wound wire, the lines did not close. By coincidence I had just read a short note in the CERN Courier<sup>6</sup> on an article by Hosoda *et al.*<sup>7</sup> They found simple wire arrangements with complicated knotted field lines. The short note convinced me that my simulation did not produce a numerical artifact.

The existence of bound but not closed field lines has been known for about 60 years,<sup>8-10</sup> but this knowledge has not found its way into textbooks. Because charged particles follow magnetic field lines to first order, complicated field lines are relevant for plasma physics (see the references in Ref. 7).

## II. NUMERICAL APPROACH

A helical wire is modeled as a polygon with short edges. The magnetic field of a single finite straight wire can be calculated exactly by integrating the Biot-Savart law along the wire,

$$\vec{B} = \frac{\mu_0 I}{4\pi} \frac{\vec{L} \times \vec{r}_0}{L^2 r_0^2 - (\vec{r}_0 \cdot \vec{L})^2} \left( \frac{L^2 - \vec{r}_0 \cdot \vec{L}}{\sqrt{r_0^2 - 2\vec{r}_0 \cdot \vec{L} + L^2}} + \frac{\vec{r}_0 \cdot \vec{L}}{r_0} \right), \quad (1)$$

where  $\vec{B}$  is the magnetic flux density generated by the thin straight wire of length  $\vec{L}$  pointing in the direction of the electrical current  $I$  at the distance  $\vec{r}_0$ . The vector  $\vec{r}_0$  starts at the beginning of the wire and ends at the point where the  $\vec{B}$ -field is to be calculated. The magnetic field of the polygon is the sum of the fields of its edges.

The  $\vec{B}$ -vectors are tangent to the magnetic field lines, that is, the field lines are defined by the differential equation

$$\frac{d\vec{s}}{ds} = \frac{\vec{B}}{B}, \quad (2)$$

where  $d\vec{s}=(dx, dy, dz)$  is an infinitesimal element of the field line. Equation (2) was solved using the fourth order Runge-Kutta algorithm. Accuracy was monitored by plotting the field lines with different step sizes. Successive points of the field lines were calculated in three dimensions and projected on the screen.

The magnetic field of a helical current cannot be expressed

by elementary functions: Even a planar circle yields elliptical integrals. Because the expressions are needed in Eq. (2), an exact expression would be of little use and the polygonal approximation is reasonable. The same approach was used to simulate the magnetic field of the ATLAS Solenoid at the large hadron collider at CERN.<sup>11</sup>

## III. RESULTS

Figure 1 shows a field line of a polygonal wire, which approximates a helix with closure. The field line is not closed after one loop. Changing the semicircular closure to a different shape strongly affects the field lines, but they still do not close after one loop. Shortening or extending the helix with the same pitch (windings per length) does not close the magnetic field lines. The field lines approach simple loops if the helix is more densely wound because the helix becomes closer to rotational symmetry (a stack of circular wires has closed field lines).

Why is this effect still not widely known? Most theories approximate the helix by a stack of circles or a current sheet, thus introducing rotational symmetry.<sup>12</sup> This symmetry eliminates the effect we have mentioned. Available field simulation codes plot field strengths,  $\vec{B}$ -vectors, energy densities or contours, but not field lines.<sup>13</sup> Because the magnetic field outside the coil is in most cases unimportant and weak, it seldom is calculated. Many believe erroneously that  $\vec{\nabla} \cdot \vec{B} = 0$  implies closure of spatially bound field lines.<sup>10</sup>

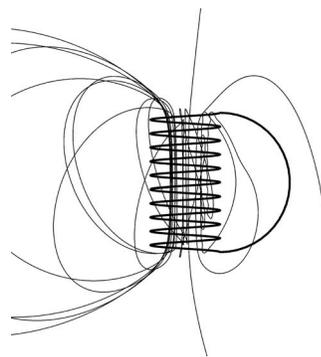


Fig. 1. A magnetic field line of a helical current with semicircular closure. The helix and semicircle are approximated by short straight lines.

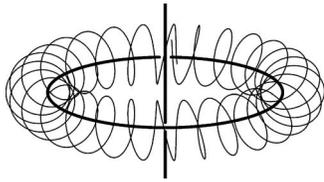


Fig. 2. A magnetic field line wrapping around a circular current with an additional infinitely long straight current along its symmetry axis.

#### IV. SIMPLE CASES

Pasta and Ulam<sup>9</sup> considered a circular current with an additional straight current along its rotational symmetry axis (see Fig. 2). From the structure of the magnetic field, it is immediately clear that a field line wrapping around the circular current is almost always open. The magnetic field lines close only after several loops or do not close at all depending on the starting point and the ratio of the two currents.

Hosoda *et al.* found chaotically knotted field lines near two circular currents at right angles.<sup>7</sup> The term “chaotic” in this context means that closely starting field lines quickly diverge. They investigated the effect of Earth’s magnetic field and of wire thickness. They argued that the effect should be observable in ordinary electronic circuits. Figure 3 shows their system with squares instead of the circles.

To speed up the calculation, I simplified the helix of Fig. 1 drastically. I ended up with a “nonplanar square” showing field lines that do not close after one loop (see Fig. 4). It came as a surprise that such a simple wire shape exhibits this behavior. Planar polygons of random shape do not show this effect: The field lines close after one loop or they do not return.

Hosoda *et al.* also found knotted field lines in a system of two separated, infinitely long straight wires at right angles with an additional constant field.<sup>7</sup> Without the small background field the field lines escape to infinity. This system is so simple that the expression for the magnetic field can be given as

$$\vec{B} = \frac{\mu_0 I}{2\pi} \frac{1}{(x-a)^2 + z^2} \begin{pmatrix} z \\ 0 \\ a-x \end{pmatrix} + \frac{\mu_0 I}{2\pi} \frac{1}{(x+a)^2 + y^2} \begin{pmatrix} -y \\ x+a \\ 0 \end{pmatrix} + \begin{pmatrix} B_{x0} \\ B_{y0} \\ B_{z0} \end{pmatrix}, \quad (3)$$

where  $2a$  is the separation of the wires. The first wire

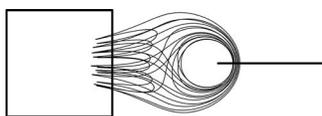


Fig. 3. Two square currents at right angles with a magnetic field line. The chaotic field line is localized in a bounded region and fills it everywhere densely.

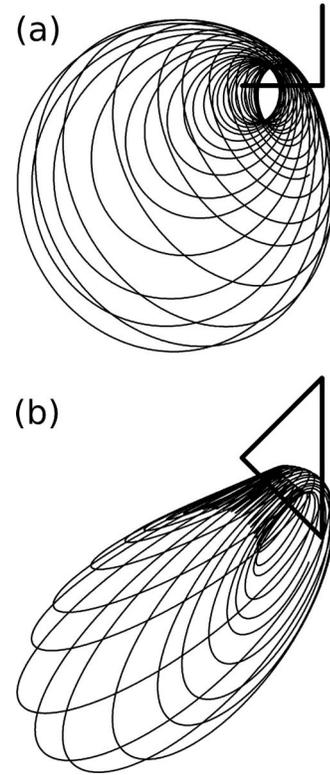


Fig. 4. A magnetic field line near a square current loop with one corner bent upward by  $90^\circ$ . (a) Front view. (b) Top view. A folded square is topologically equivalent to a circle.

( $x=a$ ) is parallel to the  $y$ -axis, and the second ( $x=-a$ ) is parallel to the  $z$ -axis. In Fig. 5 the background field is  $B_{x0} = B_{y0} = B_{z0}$ .

#### V. CONCLUSION

Textbook figures displaying magnetic field lines of a loosely wound helical coil as simple closed loops are oversimplified. Magnetic field lines of nonplanar asymmetric currents should not be expected to close after one loop. It would be better to draw the magnetic field lines as open curves. An extended discussion is given in Ref. 8.

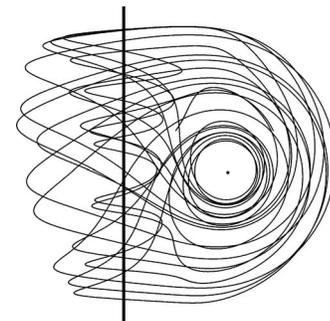


Fig. 5. A magnetic field line around two long straight wires at right angles with a small constant background field (view parallel to the wire on the right represented as a small filled circle). The magnetic field is very easy to calculate making this case accessible to undergraduate students.

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<sup>13</sup>See, for example, Maxwell<sup>®</sup>, ([ansoft.com](http://ansoft.com)).



Dynamo and Motor Set. From the 1929 catalogue of the Chicago Apparatus Company: Consists of one Little Hustler Dynamo and one Little Hustler Motor mounted on a common base and connected [with a missing belt] as a Motor-Generator Set, so that the power developed by the motor is used to drive the generator, thus affording a nice illustration of the conversion of mechanical to electric energy \$6.00. The apparatus is in the collection of Richard Zitto. (Notes by Thomas B. Greenslade, Jr., Kenyon College)